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Quantal time asymmetry: mathematical foundation and physical interpretation

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Abstract

For a quantum theory that includes exponentially decaying states and Breit–Wigner resonances, which are related to each other by the lifetime–width relation $\tau = \frac{\hbar}{\Gamma}$, where τ is the lifetime of the decaying state and Γ is the width of the resonance, one has to go beyond the Hilbert space and beyond the Schwartz–Rigged Hilbert Space $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ of the Dirac formalism. One has to distinguish between prepared states, using a space $\Phi_{-} \subset \mathcal{H}$, and detected observables, using a space $\Phi_{+} \subset \mathcal{H}$, where $-(+)$ refers to analyticity of the energy wavefunction in the lower (upper) complex energy semiplane. This differentiation is also justified by causality: a state needs to be prepared first, before an observable can be measured in it. The axiom that will lead to the lifetime–width relation is that Φ_{+} and Φ_{-} are Hardy spaces of the upper and lower semiplane, respectively. Applying this axiom to the relativistic case for the variable $\mathsf{s} = p_{\mu} p^{\mu}$ leads to semigroup transformations into the forward light cone (Einstein causality) and a precise definition of resonance mass and width.

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1. Time asymmetry

Time asymmetry does *not* mean *time reversal non-invariance*, i.e. it is not described by a Hamiltonian H that does not commute with the time reversal A_T or the CP operator [1]:

$$[A_T, H] \neq 0 \qquad \text{or} \qquad [CP, H] \neq 0. \tag{1}$$

It may be intriguing, however, to speculate about a possible connection [2].

Time asymmetry does *not* mean *irreversibility* or thermodynamic arrow of time. It is not entropy increase in an isolated classical system, $\frac{dS}{dt} > 0$. There may, however, be some relation to entropy increase. For instance, Peierls (1979) explains entropy increase from initial boundary conditions in Boltzmann's Stosszahl ansatz [3].

Time asymmetry also does *not* mean the usual quantum mechanical arrow of time for 'open' quantum systems, brought about by an external reservoir [4] described by the Liouville equation,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = LW(t) \equiv -\mathrm{i}[H, W(t)] + \mathcal{I}W(t), \tag{2}$$

in which a 'superoperator', $\mathcal{I}$, describes the external effect of the reservoir or of a measurement apparatus or of another environment (decoherence).

By time asymmetry, we mean *time asymmetric boundary conditions* for time symmetric dynamical equations.

The best-known example is the radiation arrow of time [5]. Maxwell's equations (dynamical differential equations) are symmetric in time. A boundary condition excludes the strictly incoming fields and selects only the retarded fields of the other sources in the region:

$$A^\mu(x) = A^\mu_{\text{ret}}(x) + A^\mu_{in}(x) = A^\mu_{\text{ret}}(x), \tag{3}$$

$$A^\mu_{in}(x) = 0 \qquad \text{(Sommerfeld radiation condition).} \tag{4}$$

The boundary condition is an additional 'principle of nature' that chooses, of the two solutions of the Maxwell equations,

$$A^\mu_\mp(\vec{x}, t) = \int \delta\left(t' - \left(t \mp \frac{|\vec{x} - \vec{x}'|}{c}\right)\right) \frac{j^\mu(\vec{x}', t')}{|\vec{x} - \vec{x}'|}\, \mathrm{d}^3x'\, \mathrm{d}t', \tag{5}$$

the retarded solution

$$A^\mu_{\text{ret}}(\vec{x}, t) \equiv A^\mu_-(\vec{x}, t) = \int \frac{j^\mu\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|}\, \mathrm{d}^3x'. \tag{6}$$

The 'disturbance' $A^\mu(x)$ at the position $\vec{x}$ at time t is caused by the source j^μ at another point $\vec{x}'$, at an earlier time $t' = t - \frac{|\vec{x}-\vec{x}'|}{c} \leqslant t$. Radiation must be emitted (at t') by a source before it can be detected by a receiver at $t \geqslant t'$.

2. Dynamical equations and their boundary conditions

Standard quantum mechanics lacks time asymmetric boundary conditions. The dynamical equations are as follows.

In the Heisenberg picture,

$$\mathrm{i}\hbar \frac{\partial \Lambda(t)}{\partial t} = -[H, \Lambda(t)] \tag{7a}$$

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi(t) = -H\psi(t) \tag{7b}$$

for the observable $\Lambda(t) = |\psi(t)\rangle\langle\psi(t)|$, with state $W = |\phi\rangle\langle\phi|$ kept fixed.

Or *in the Schrödinger picture*,

$$\mathrm{i}\hbar \frac{\partial W(t)}{\partial t} = [H, W(t)] \tag{8a}$$

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\phi(t) = H\phi(t) \tag{8b}$$

for the state $W(t)$ or $\phi(t)$ with observable $\Lambda = |\psi\rangle\langle\psi|$ kept fixed.

In standard quantum mechanics [6], one solves these equations under the *boundary conditions*

$$\text{set of states } \{\phi\} = \mathcal{H} = \text{Hilbert space} = \text{set of observables } \{\psi\}. \tag{9}$$

As a consequence of the Hilbert space boundary condition, one obtains from the Stone–von Neumann theorem [7]

$$\psi(t) = \mathrm{e}^{\mathrm{i}Ht}\psi, \qquad -\infty < t < +\infty \qquad \text{in the Heisenberg picture} \tag{10}$$

$$\phi(t) = \mathrm{e}^{-\mathrm{i}Ht}\phi, \qquad -\infty < t < +\infty \qquad \text{in the Schrödinger picture.} \tag{11}$$

The sets of operators

$$\{U(t) = \mathrm{e}^{\mathrm{i}Ht} : -\infty < t < +\infty\} \tag{12}$$

and

$$\{U^\dagger(t) = \mathrm{e}^{-\mathrm{i}H^\dagger t} : -\infty < t < +\infty\} \tag{13}$$

form a group, because the products $U(t_1)U(t_2) = U(t_1+t_2)$ and the inverses $U^{-1}(t) = U(-t)$ exist. For every observable $\psi(t) = U(t)\psi$ at time t there is also an observable $\psi(-t) = U^{-1}(t)\psi$ at time $-t$. The same applies for the state ϕ and the operator $U^\dagger(t)$.

In quantum physics one distinguishes [8] between *states*, which are described by density operators W or by state vectors ϕ, and *observables*, which are described by operators $A = A^\dagger$, $\Lambda = \Lambda^2$ or by observable vectors ψ if $\Lambda = |\psi\rangle\langle\psi|$.

State W (*in*-states ϕ^+ of scattering experiment) *is prepared by* a preparation apparatus (e.g. accelerator).

Observable A (*out*-observables ψ^-, 'out-state') *is registered by* a registration apparatus (e.g. detector).

Experimental quantities, $\mathcal{P}_W(\Lambda(t))$, are the probabilities to measure the observable Λ in the state W. They are calculated in theory as Born probabilities. They are measured as ratios of large numbers of detector counts ('relative frequencies').

$$\mathcal{P}_W(\Lambda(t)) \equiv \mathrm{Tr}(\Lambda(t)W_0) = \mathrm{Tr}(\Lambda_0 W(t)) \approx N(t)/N \tag{14}$$

$$\underset{\text{in the Heisenberg picture}}{|\langle\psi(t)|\phi\rangle|^2} = \underset{\text{in the Schrödinger picture}}{|\langle\psi|\phi(t)\rangle|^2} \tag{15}$$

The comparison between theory and experiment is given by

$$\mathcal{P}_W(\Lambda(t)) \approx \frac{N(t)}{N}. \tag{16}$$

The left-hand side is the calculated prediction, and the right-hand side is the ratio of detector counts, where $N(t)$ and N are 'large' integers. The comparison between theory and experiment is indicated by $\approx$.

What is the experimental evidence for this time evolution *group*? It is obvious that a state ϕ must be prepared before the observable $|\psi(t)\rangle\langle\psi(t)|$ can be measured in it (causality), e.g. the detector cannot register the decay products before the decaying state has been prepared. This means that we have a *quantum mechanical arrow of time.*

The Born probability to measure the observable $|\psi(t)\rangle\langle\psi(t)|$ in the state ϕ,

$$\frac{N(t)}{N} \approx \mathcal{P}_\phi(\psi(t)) = |\langle\psi(t)|\phi\rangle|^2 = |\langle \mathrm{e}^{\mathrm{i}Ht}\psi|\phi\rangle|^2 = |\langle\psi|\mathrm{e}^{-\mathrm{i}Ht}\phi\rangle|^2 = |\langle\psi|\phi(t)\rangle|^2, \tag{17}$$

exists (experimentally) only for $t \geqslant t_0\ (=0)$, where t_0 is the preparation time of the state ϕ. In contrast, the unitary group of the Hilbert space axiom predicts $|\langle\psi(t)|\phi\rangle|^2$ for all $-\infty < t < +\infty$.

As a consequence of this obvious phenomenological condition (causality), one obtains the quantum mechanical arrow of time.

In the Heisenberg picture the time translated observables

$$\psi(t) = \mathrm{e}^{\mathrm{i}Ht}\psi \text{ are physically defined only for } t > t_0 = 0. \tag{18}$$

In the Schrödinger picture the time-evolved states

$$\phi(t) = \mathrm{e}^{-\mathrm{i}Ht}\phi \text{ are physically defined only for } t > t_0 = 0. \tag{19}$$

The time evolution is asymmetric, $0 \leqslant t < \infty$, and given by the semigroup

$$\mathcal{U}^\times(t) = \mathrm{e}^{-\mathrm{i}H^\times t} \qquad \text{with} \qquad 0 \leqslant t < \infty \text{ for the states } \phi \text{ or } W \tag{20}$$

or by the semigroup

$$\mathcal{U}(t) = \mathrm{e}^{\mathrm{i}Ht} \qquad \text{with} \qquad 0 \leqslant t < \infty \text{ for the observables } \psi \text{ or } \Lambda. \tag{21}$$

Therefore we have the task: find a theory (for instance choosing new boundary conditions to replace the Hilbert space axiom) for which the solutions of the Schrödinger equation are given by the *semigroup* $\mathcal{U}^\times(t)$ (20) and for which the solutions of the Heisenberg equation are given by the *semigroup* $\mathcal{U}(t)$ (21).

Remark (semigroup symmetries of spacetime)**.** In standard quantum mechanics, time evolution is a subgroup of the spacetime symmetry transformations, which, according to the Wigner–Bargmann theorem, are represented by the (projective) unitary representations in $\mathcal{H}$ of the Galilei group G (for non-relativistic spacetime) and the Poincarè group $\mathcal{P}$ (for relativistic spacetime) [9].

In the RHS formulation [10–12] using the Schwartz space Gelfand triplet, $\Phi \subset \mathcal{H} \subset \Phi^\times$, with the Dirac basis vector expansion

$$\phi = \int \mathrm{d}\lambda |\lambda\rangle\langle\lambda|\phi\rangle \text{ for } \phi \in \Phi, \qquad \text{with} \qquad |\lambda\rangle = |\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\rangle \in \Phi^\times, \tag{22}$$

symmetry transformations $g \in G$ are represented by a triplet of operators [12–14]

$$\begin{array}{llll} & U_\Phi(g^{-1}) \subset & U(g^{-1}) = U^\dagger(g) & \subset U^\times(g) \\ \text{acting on} & \Phi \subset & \mathcal{H} \quad \mathcal{H} & \subset \Phi^\times \\ \text{e.g. for time translations} & \mathrm{e}^{-\mathrm{i}Ht}|_\Phi \subset & \mathrm{e}^{-\mathrm{i}Ht} = (\mathrm{e}^{\mathrm{i}Ht})^\dagger & \subset \mathrm{e}^{-\mathrm{i}H^\times t}, \quad -\infty < t < \infty. \end{array}$$

Here $U(g)$ is a unitary representation of g in $\mathcal{H}$. $U_\Phi(g^{-1})$ is the restriction of $U(g^{-1})$ to the dense subspace $\Phi \subset \mathcal{H}$, which is a continuous operator with respect to the Φ topology, and $U^\times(g)$ is the conjugate operator of $U_\Phi(g^{-1})$ in $\Phi^\times$, defined by

$$\langle\phi|U^\times(g)|\lambda\rangle = \langle U_\Phi(g^{-1})\phi|\lambda\rangle, \qquad \text{for all} \qquad g \in G, \phi \in \Phi, |\lambda\rangle \in \Phi^\times. \tag{23}$$

The algebra of observables H, J_i, P_i, etc is obtained by deriving $U(g)$ using the limits with respect to the Φ-topology; they are continuous operators in Φ. Their conjugate operators $H^\times$, $J_i^\times$, etc are continuous in $\Phi^\times$ [13, 15].

This is a beautiful theory [10, 12–14], but it assumes that for every transformation $g \in G$ of the observable relative to the state, $g : \psi \to \psi^g$, there exists an inverse transformation *also of the observable relative to the state* (not the state relative to the observable), $g^{-1} : \psi \to \psi^{g^{-1}}$. For rotations and boosts this is meaningful, but for time translation, it would give an answer to the question: what was the probability of an observable ψ in a state at a time $t_0 + t$, with $t < 0$, before the state will be prepared at t_0? This contradicts causality. Therefore one must find a space Φ_+ for the observables $\{\psi\}$ such that the Galilei transformations of non-relativistic spacetime are represented by a semigroup $U_+(R, t, \mathbf{x}, \mathbf{v}), t \geqslant 0$. Similarly, for the relativistic case mentioned in section 8, one must find a space (which we will also call Φ_+) in which the transformations of the detected observables relative to the prepared state form only a semigroup into the forward light cone [16] expressing Einstein causality:

$$P_+ = \{(\Lambda, x) : (\Lambda^0_0 \geqslant 1, \det \Lambda = +1), x^2 = t^2 - \mathbf{x}^2 \geqslant 0, t \geqslant 0\}. \tag{24}$$

3. Resonances and decay

To find the theory that provides the time asymmetry semigroup, we use scattering, resonance and decay phenomena [17]. Resonances and decaying states are characterized by definite values of the discrete quantum numbers such as charges (particles species label) and by angular momentum j. In addition they are defined by two real numbers. Resonances are characterized by energy E_R and width Γ, and decaying states are characterized by energy E_D and lifetime τ. Their properties are contrasted as follows:

(E_R, Γ) defined by Breit–Wigner (Lorentzian) scattering amplitude	$\left(E_D, R = \frac{1}{\tau}\right)$defined by exponential partial decay rate
$a_j^{BW} = \frac{r_\eta}{E-(E_R-\mathrm{i}\frac{\Gamma}{2})}; 0 \leqslant E < \infty$	$R_\eta(t) = \frac{d}{\mathrm{d}t} P_\eta(t), R = \sum_\eta R_\eta(0)$ $R_\eta(t) = R_\eta(0)\,\mathrm{e}^{-t/\tau} = R_\eta(0)\,\mathrm{e}^{-Rt}$ where probability $P_\eta(t) = \lvert\langle\psi_\eta\vert\phi_D(t)\rangle\rvert^2$
Resonances appear in scattering, e.g. $\mathrm{e}^+\mathrm{e}^- \rightarrow Z \rightarrow \mathrm{e}^+\mathrm{e}^-$	Decaying states are observed in decay, e.g. $\pi^- p \rightarrow \Lambda K^0_S$ $\quad\quad\quad\quad \lfloor\!\longrightarrow \pi^+\pi^-$
Resonances are measured by Breit–Wigner line shape in the cross section	Decaying states are measured by the exponential law for the counting rate of the decay products η
$\sigma_j(E) \sim \lvert a_j(E)\rvert^2 = \left\lvert\frac{r_\eta}{E-(E_R-\mathrm{i}\frac{\Gamma}{2})} + B(E)\right\rvert^2$ $B(E)$ is a slowly varying function of E, the background.	$R_\eta(t) \approx \frac{\Delta N(t_i)}{\Delta t_i} \propto \mathrm{e}^{-\frac{t}{\tau}}$ where $\Delta N(t_i)$ is the number of decay products registered in the detector during the time interval Δt_i around t_i.
An example of how well this agrees with an experiment is given in figure 1.	An example of the accuracy of this experimental law is given in figure 2.

Many people think that

$$\{\text{resonances}\} \equiv \{\text{decaying states}\}, \tag{25}$$

and especially for non-relativistic quantum mechanics, a common assumption is that

$$\frac{\hbar}{\Gamma} = \tau \quad \left(\text{or at least } \frac{\hbar}{\Gamma} \approx \tau\right). \tag{26}$$

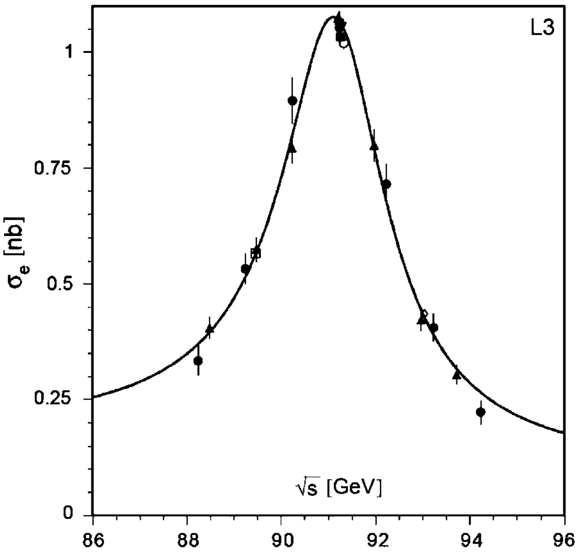

Figure 1. Breit–Wigner for the Z-boson resonance [18].

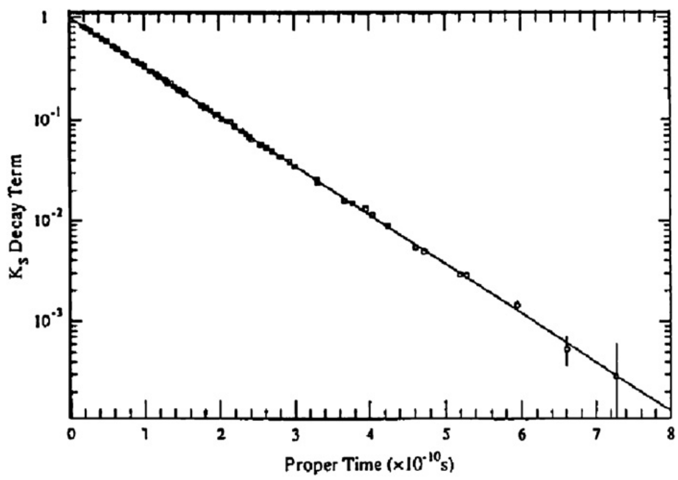

Figure 2. Exponential for the K_S^0 decay rate [19].

This relation is based on the Weisskopf–Wigner (WW) approximation [20], and in standard quantum theory there is no proof of it, as e.g. stated by

> M. Levy: [21] '... There does not exist... a rigorous theory to which these various (WW) methods can be considered as approximations.'

The energy of a resonance is the complex number

$z_R = E_R - \mathrm{i}\frac{\Gamma}{2}$, the pole position of the S matrix or of the scattering amplitude $a_j(E)$.

What one can show using the Weisskopf–Wigner approximation methods is that the probability of a prepared resonance state ϕ with Breit–Wigner energy distribution of width Γ is obtained as [22]

$$\mathcal{P}_{\phi(t)}(\psi) \sim \mathrm{e}^{-\Gamma t/\hbar} + \Gamma \times (\text{additional terms}). \tag{27}$$

Thus, in the Weisskopf–Wigner approximation, the probability rate has a non-exponential term proportional to the width Γ. In general, one can prove that there is no Hilbert space vector $\phi(t)$ that obeys the exact exponential law [23]. If, as in figure 2 for the K_S^0-decay experiment, the time dependence of $|\langle\psi_{\pi\pi}|\phi^{K_S^0}(t)\rangle|^2 = \frac{N(t_i)}{N}$ is to be a perfect exponential (as shown by this experiment [19]) then the decaying state vector $\phi^{K_S^0}$ cannot be a Hilbert space vector.

4. Complex energies and a beginning of time

The simplest way to derive an exactly exponential decay probability is to postulate a state vector ϕ^G, which has the property

$$H\phi^G = \left(E_R - \mathrm{i}\frac{\Gamma}{2}\right)\phi^G \qquad \text{and} \qquad \phi^G(t) = \mathrm{e}^{-\mathrm{i}Ht}\phi^G, \tag{28}$$

and whose decay probability into any observable $|\psi\rangle\langle\psi|$ is

$$\begin{aligned}\mathcal{P}_{\phi^G(t)} &= |\langle\psi|\phi^G(t)\rangle|^2 = |\langle\psi|\,\mathrm{e}^{-\mathrm{i}Ht}|\phi^G(t)\rangle|^2\\ &= |\langle\psi|\phi^G\rangle\,\mathrm{e}^{-\mathrm{i}(E_R-\mathrm{i}\Gamma/2)t}|^2 = |\langle\psi|\phi^G\rangle|^2\,\mathrm{e}^{-\Gamma t}.\end{aligned} \tag{29}$$

This vector ϕ^G of Gamow [24] makes no sense in standard quantum mechanics because

(i) it has a complex eigenvalue for a self-adjoint H, $\phi^G = |E_R - \mathrm{i}\Gamma/2\rangle$;
(ii) it leads to the 'exponential catastrophe' [25], because in standard quantum mechanics the time extends from $-\infty$: $-\infty < t < \infty$.

To define a vector with the properties (28) and (29), one needs a theory for which

(i) the energy extends to values in the complex plane;
(ii) the time is restricted to $0 \leqslant t < +\infty$ (because a K_S^0 must be prepared first before one can detect its decay products $\pi^+\pi^-$, and to avoid the exponential catastrophe).

In the standard (Hilbert space) theory of quantum physics, the time t has the values $-\infty < t < +\infty$, and the energy E is real (spectrum of self-adjoint Hamiltonian H) and bounded from below (stability of matter): $-\infty \neq E_0 \leqslant E < \infty$.

Nevertheless, one speaks of complex energy:

(i) for the analytic S matrix: $S_j(E) \to S_j(z)$;
(ii) for the Gamow states: $z_R = E_R - \mathrm{i}\Gamma/2$;
(iii) for the Lippmann–Schwinger equation or in the propagator of field theory: $z = E \pm \mathrm{i}\epsilon$, ϵ infinitesimal.

Thus experiments require a quantum theory in which

(i) the time t has a 'preferred direction': $t_0 = 0 \leqslant t < \infty$;
(ii) the energy E can take complex (discrete and continuous) values in the complex planes: $E \to z \in \mathbb{C}_\pm$:

$$|E\rangle \longrightarrow |E \pm \mathrm{i}\epsilon\rangle \longrightarrow |z^\pm\rangle \qquad z \in \mathbb{C}_\pm. \tag{30}$$

The conclusion from this is that one has to restrict the set of allowed energy wavefunctions $\phi(E) = \langle^+E|\phi^+\rangle$, which in standard quantum theory in $\mathcal{H}$ are represented by a *class* of Lebesgue square integrable functions, to a smaller set.

The first step in this direction was already taken by Schwartz, inspired by the Dirac bra–ket formalism, when he restricted the set of *classes* of Lebesgue square integrable functions $\mathcal{L}^2 = \{\{h(E)\}\}$ to the set of smooth, rapidly decreasing (Schwartz) functions $\{\phi(E)\}$. This replacement of the Hilbert space $\mathcal{H}$ by the Schwartz space Φ gave a mathematical meaning to the Dirac formalism [26].

The second step is to restrict the set of Schwartz space functions $\phi(E)$ further to the set of smooth functions that have analytic extensions in the upper and lower complex energy semiplanes, $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively:

$$\{\overline{\phi(E)} = \langle\phi|E\rangle\} \to \begin{cases} \langle\phi^+|E+\mathrm{i}\epsilon\rangle \equiv \langle\phi^+|E^+\rangle \to \langle\phi^+|z^+\rangle; & z\in\mathbb{C}_+ \\ \langle\psi^-|E-\mathrm{i}\epsilon\rangle \equiv \langle\psi^-|E^-\rangle \to \langle\psi^-|z^-\rangle; & z\in\mathbb{C}_- \end{cases}. \tag{31}$$

5. Rigged Hilbert space

The mathematical basis for these modifications (31) is rigged Hilbert spaces (RHS) or Gelfand triplets. The RHS was not devised for time asymmetry or the theory of resonances and decay, but to provide a mathematical justification for Dirac's bra- and ket-formalism [26].

Step 1. The first step away from the Hilbert space was to get the Dirac formalism, which has the following two properties:

(i) The solutions of both the Heisenberg and the Schrödinger equations (for observables and states) have a Dirac basis vector expansion,

$$\phi = \sum_{j,j_3}\sum_n |E_n, j, j_3)(E_n, j, j_3|\phi\rangle + \sum_{j,j_3,\eta}\int \mathrm{d}E|E, j, j_3, \eta\rangle\langle E, j, j_3, \eta|\phi\rangle, \tag{32}$$

which is the analogue of $\vec{x} = \sum_{i=1}^3 \vec{e}_i x^i$, and where the basis vectors $|E, j, j_3, \eta\rangle \equiv |E\rangle$ are 'eigenkets' of H (and of a complete system of commuting observables for the other quantum numbers j, j_3, η). The basis vectors in (32) are defined as[1] the eigenkets of an operator H

$$\langle H\phi|E, j, j_3, \eta\rangle = \langle\phi|H^\times|E, j, j_3, \eta\rangle = E\langle\phi|E, j, j_3, \eta\rangle \qquad \text{for all} \quad \phi\in\Phi, \tag{33a}$$

or as ordinary eigenvectors

$$H|E_n, j, j_3, \eta) = E_n|E_n, j, j_3\eta). \tag{33b}$$

In (32) and (33*a*), the values of E are from a continuous set, e.g., $0 \leqslant E < \infty$.

(ii) The coordinates or 'scalar products' or the bra–kets $\langle E|\phi\rangle = \phi(E)$ are smooth, rapidly decreasing functions of E ('Schwartz function' $\mathcal{S}_{\mathbb{R}_+}$). One gets a triplet of function spaces

$$\{\phi(E)\} = \{\psi(E)\} = \mathcal{S}_{\mathbb{R}_+} \subset L^2 \subset \mathcal{S}^\times \tag{34}$$

and corresponding to this a triplet of abstract vector spaces

$$\{\phi\} = \{\psi\} = \Phi \subset \mathcal{H} \subset \Phi^\times \ni |E\rangle. \tag{35}$$

The space $\Phi^\times$ denotes the space of continuous antilinear functionals of the space Φ; for the Hilbert space, $\mathcal{H}^\times = \mathcal{H}$. Dirac kets are antilinear continuous Schwartz space functionals.

[1] The first equality defines the conjugate operator $H^\times$ of H, $H^\times \supset H^\dagger$.

The triplets (34) and (35) constitute rigged Hilbert spaces. (Φ has a locally convex nuclear topology such that the Dirac basis vector expansion (32) is the nuclear spectral theorem.) The RHS (34) of functions and distributions is equivalent (algebraically and topologically) to the abstract RHS (35).

This Schwartz space triplet (of axiomatic quantum field theory) will not provide time asymmetry, because solutions of the dynamical (Schrödinger or Heisenberg) equation with boundary condition $\phi \in \Phi$, $\psi \in \Phi$ (Schwartz space) are again given by a group [15]

$$\phi(t) = \mathrm{e}^{-\mathrm{i}Ht}\phi(0), \qquad \psi(t) = \mathrm{e}^{\mathrm{i}Ht}\psi(0), \qquad -\infty < t < +\infty. \tag{36}$$

For the Schwartz–Dirac kets one thus also obtains [15]

$$\mathrm{e}^{-\mathrm{i}H^{\times}t}|E\rangle = \mathrm{e}^{-\mathrm{i}Et}|E\rangle \qquad -\infty < t < +\infty. \tag{37}$$

This means Step 1 away from the Hilbert space provides the Dirac kets (33*a*) and the Dirac basis vector expansion (32), but the Dirac formalism still does not lead to complex energy eigenvalues and resonance states (28), (29).

Step 2. Because observables $|\psi^-\rangle\langle\psi^-|$ defined by the detector and states ϕ^+ defined by the preparation apparatus are physically different entities, one should also distinguish mathematically between a *space of observables*, which we call Φ_+,

$$\Phi_+ \ni \psi^- = \sum_{j,j_3,\eta}\int_0^{\infty} \mathrm{d}E|E, j, j_3, \eta^-\rangle\langle^- E, j, j_3, \eta|\psi^-\rangle, \tag{38}$$

and a *space of states*, which we call Φ_-,

$$\Phi_- \ni \phi^+ = \sum_{j,j_3,\eta}\int_0^{\infty} \mathrm{d}E|E, j, j_3, \eta^+\rangle\langle^+ E, j, j_3, \eta|\phi^+\rangle. \tag{39}$$

The two Dirac basis vector expansions (38), (39) use different kinds of Dirac kets, which are denoted as

$$|E, j, j_3, \eta^{\mp}\rangle \in \Phi_{\pm}^{\times}. \tag{40}$$

This is suggested by the Lippmann–Schwinger in-plane waves $|E^+\rangle$ for the prepared in-states ϕ^+, and the Lippmann–Schwinger out-plane waves $|E^-\rangle$ for the detected out-observables ψ^-.[2]

Because of the $+\mathrm{i}\epsilon$ in the Lippmann–Schwinger equation, the energy wavefunction of the prepared in-state ϕ^+,

$$\phi^+(E) = \langle^+ E, j, j_3, \eta|\phi^+\rangle = \overline{\langle\phi^+|E, j, j_3, \eta^+\rangle}, \tag{41}$$

is the boundary value of an analytic function in the lower complex energy semiplane (for complex energy $z = \overline{E + \mathrm{i}\epsilon} = E - \mathrm{i}\epsilon$ immediately below the real axis 2nd sheet of the S-matrix $S_j(z)$). Similarly, one surmises that the energy wavefunction of the observable $|\psi^-\rangle\langle\psi^-|$,

$$\psi^-(E) \equiv \langle^- E, j, j_3, \eta|\psi^-\rangle, \tag{42}$$

extends to an analytic function in the upper complex energy plane. Thus one conjectures that energy wavefunctions can only be those Schwartz functions that can be analytically continued into the lower or upper complex energy semiplane, respectively. The space of allowed state vectors must be smaller than the Schwartz space, and the space of allowed kets will be larger than the set of Dirac kets and thus can include the eigenkets of complex energy.

[2] The only major alteration of the standard S-matrix formalism is that here we use out-observables fulfilling the Heisenberg equation and not out-states fulfilling the Schrödinger equation.

6. Conjecturing the Hardy space axiom of time asymmetric quantum mechanics

To conjecture the spaces of ϕ^+ and of ψ^-, we start with the definition of a resonance by an S-matrix pole at $z_R = E_R - \mathrm{i}\Gamma/2$. If we derive from it

a Breit–Wigner resonance amplitude and relate to this Breit–Wigner a Gamow state vector

$$a_j^{BW_i} = \frac{R_i}{E - z_{R_i}} \quad \Longleftrightarrow \quad \phi_j^G = |z_{R_i}, j, j_3, \eta^-\rangle\sqrt{2\pi\Gamma_i} = \int \mathrm{d}E |E, j, j_3, \eta^-\rangle \frac{i\sqrt{\frac{\Gamma}{2\pi}}}{E - z_R}$$

$$\text{for every pole } z_{R_i} \qquad z_{R_i} = E_{R_i} - \mathrm{i}\Gamma_i/2 \tag{43}$$

which has the properties:

1. It is an eigenket with a discrete complex eigenvalue (as Gamow wanted) of the Hamiltonian:

$$H^\times |E_R - \mathrm{i}\Gamma/2^-\rangle = (E_R - \mathrm{i}\Gamma/2)|E_R - \mathrm{i}\Gamma/2^-\rangle, \qquad |E_R - \mathrm{i}\Gamma/2^-\rangle \in \Phi_+^\times. \tag{44}$$

2. It has the time evolution

$$\begin{aligned}
&\langle \mathcal{U}(t)\psi_\eta^- | \phi_j^G\rangle = \langle \psi_\eta^- | \mathcal{U}^\times(t)\phi_j^G\rangle \sim \\
&\langle \mathrm{e}^{\mathrm{i}Ht/\hbar}\psi_\eta^- | E_R - \mathrm{i}\Gamma/2^-\rangle = \langle \psi_\eta^- | \mathrm{e}^{-\mathrm{i}H^\times t/\hbar} | E_R - \mathrm{i}\Gamma/2^-\rangle \\
&\qquad = \mathrm{e}^{-\mathrm{i}E_R t/\hbar}\mathrm{e}^{-(\Gamma/2)t/\hbar}\langle \psi_\eta^- | E_R - \mathrm{i}\Gamma/2^-\rangle \quad \text{for all} \quad \psi_\eta^- \in \Phi_+,
\end{aligned} \tag{45}$$

then we will show that

$$\text{a resonance of width } \Gamma \equiv \text{a decaying state with lifetime } \tau = \frac{\hbar}{\Gamma}, \tag{46}$$

and we will have a theory that unites resonance scattering and exponential decay.

To derive these results, further conditions had to be put on the analytic wavefunctions $\phi^+(z)$ and $\overline{\psi^-(z)}$ in the lower complex semiplane 2nd sheet. These conditions suggested Hardy functions (H Baumgartel).

Thus one is led to a *new hypothesis* [17]:

The energy wavefunctions of a state are smooth Hardy functions analytic on the lower complex semiplane $\mathbb{C}_-$:

$$\phi^+(E) = \langle {}^+E|\phi^+\rangle \in (\mathcal{H}_-^2 \cap \mathcal{S})_{\mathbb{R}_+}. \tag{47}$$

The energy wavefunctions of an observable are smooth Hardy functions analytic on the upper complex semiplane $\mathbb{C}_+$:

$$\psi^-(E) = \langle {}^-E|\psi^-\rangle \in (\mathcal{H}_+^2 \cap \mathcal{S})_{\mathbb{R}_+}. \tag{48}$$

This hypothesis is not so far off from the properties that one often used for mathematical manipulations in scattering theory. This new hypothesis led to the construction of the two Hardy space triplets [27], i.e. two RHSs from which the two Dirac basis vector expansions (38), (39) follow as the nuclear spectral theorem [28].

Therewith we have arrived at a new axiom of quantum mechanics, which is the *Hardy axiom space*.

The set of prepared (in-) states defined by the preparation apparatus (e.g. accelerator) is represented by

$$\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times. \tag{49}$$

The set of (out-) observables defined by the registration apparatus (e.g. detector) is represented by

$$\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times. \tag{50}$$

Here $\Phi_\mp$ are Hardy spaces of the semiplanes $\mathbb{C}_\mp$. The kets $|z^\pm\rangle$, $|E^\pm\rangle$, $|E_R - \mathrm{i}\Gamma/2^\pm\rangle$ are functionals on the space $\Phi_\mp$. In particular, the exponentially decaying Gamow ket $|E_R - \mathrm{i}\Gamma/2^-\rangle$ is a functional on the space Φ_+.

Experimentalists distinguish between preparation apparatus (accelerator) and registration apparatus (detector). In the foundations of quantum mechanics, one talks of states and observables as different entities [8, 29]. In Hilbert space theory one identifies them mathematically. The Hardy space axiom also distinguishes mathematically between states, $\phi^+ \in \Phi_-$, and observables, $\psi^- \in \Phi_+$, as different dense subspaces of the same Hilbert space $\mathcal{H}$. This is entirely natural.

7. Semigroup time evolution

The dynamical equations of quantum mechanics (7*a*) or (8*b*) can now be solved for the observable ψ^- and the state ϕ^+ under the Hardy space boundary condition $\psi^- \in \Phi_+$ and $\phi^+ \in \Phi_-$. As a consequence of the Paley–Wiener theorem [30] it follows that the solutions of dynamical equations of observables in space Φ_+ (Heisenberg equation) and of states in the space Φ_- (Schrödinger equation) are given by the semigroups

$$\psi^-(t) = \mathrm{e}^{\mathrm{i}Ht}\psi^- \qquad t_0 = 0 \leqslant t < \infty \tag{51}$$

$$\phi^+(t) = \mathrm{e}^{-\mathrm{i}Ht}\phi^+ \qquad t_0 = 0 \leqslant t < \infty, \tag{52}$$

respectively.

The Lippmann–Schwinger scattering states in $\Phi_+^\times$ fulfil

$$\mathrm{e}^{-\mathrm{i}H^\times t}|Ej\eta^-\rangle = \mathrm{e}^{-\mathrm{i}Et}|Ej\eta^-\rangle \qquad 0 \leqslant t < \infty. \tag{53}$$

Therefore, the probability for the time-evolved observable $\psi^-(t)$ in the state ϕ^+ can now be calculated:

$$\mathcal{P}_{\phi^+}(\psi^-(t)) = |\langle\psi^-(t)|\phi^+\rangle|^2 = |\langle \mathrm{e}^{\mathrm{i}Ht}\psi^-|\phi^+\rangle|^2 = |\langle\psi^-|\mathrm{e}^{-\mathrm{i}H^\times t}\phi^+\rangle|^2 \quad \text{for} \quad t \geqslant t_0 = 0 \text{ only.} \tag{54}$$

This is in agreement with the phenomenological conclusion at which we arrived in (17) on the basis of causality. For the detector counts, this means that

$$\mathcal{P}_{\phi^+}(\psi^-(t)) \sim N(t)/N \quad \text{can be measured only for } t \geqslant 0, \tag{55}$$

i.e., after the state has been prepared.

As a special case, the probability for the decay products $|\psi_\eta^-\rangle\langle\psi_\eta^-|$ in a Gamow state $\phi^G = |E_R - \mathrm{i}\Gamma/2^-\rangle$ is predicted to be

$$|\langle \mathrm{e}^{\mathrm{i}Ht/\hbar}\psi_\eta^-|\phi^G\rangle|^2 = \mathrm{e}^{-\Gamma t/\hbar}|\langle\psi_\eta^-|\phi^G\rangle|^2 \quad \text{only for } t \geqslant 0. \tag{56}$$

This avoids the 'exponential catastrophe' [25] for Gamow states.

8. Application: correct values of mass and width of the *Z*-boson and other relativistic resonances

Causal evolution of the non-relativistic spacetime can be extended to relativistic spacetime. Whereas non-relativistic time evolution is described by the Galilei group with invariants m, E, j, η (particle species or channel number), the relativistic time evolution is described by the Poincarè group $\mathcal{P}$ with invariants $\mathsf{s} = p^\mu p_\mu$ and j, spin, for a particle species η [31]. The causal time evolution is then given by the Poincarè semigroup into the forward light cone,

Table 1. *Z*-boson masses and widths from PDG.

$M_Z = 91.1875 \pm 0.0021$ GeV	$M_R = 91.1611 \pm 0.0023$ GeV	$\overline{M}_Z = 91.1526 \pm 0.0023$ GeV
$\Gamma_Z = 2.4939 \pm 0.0024$ GeV	$\Gamma_R = 2.4943 \pm 0.0024$ GeV	$\overline{\Gamma}_Z = 2.4945 \pm 0.0024$ GeV

$$\mathcal{P}_+ = \{(\Lambda, x) : x^2 = t^2 - \boldsymbol{x}^2 \geqslant 0, t \geqslant 0\}. \tag{57}$$

In analogy to Wigner's unitary group representations $[\mathsf{s} = m^2, j]$ for stable particles, the relativistic resonance particles are described by semigroup representations into the forward light cone with invariants $[\mathsf{s}_R, j]$ [32], where s_R is a complex mass squared given by the pole position of the relativistic *S*-matrix. This means that the resonance amplitude $a_j^{\text{res}}(\mathsf{s})$ is given by

$$a_j^{\text{res}}(\mathsf{s}) = \frac{r}{\mathsf{s} - \mathsf{s}_R}. \tag{58}$$

The basis vectors of the semigroup representation $[\mathsf{s}_R, j]$ are analogous to the Wigner kets given by the relativistic Gamow ket $|[\mathsf{s}_R, j]\hat{\mathbf{p}} j_3^-\rangle$ [32].

With only the complex number for the pole position s_R given, there can be many parameterizations in terms of two real parameters, (M, Γ). For instance, for the *Z*-boson one used

$$\mathsf{s}_R = (M_R - \mathrm{i}\Gamma_R/2)^2 \tag{59a}$$
$$\mathsf{s}_R = \overline{M}_Z^2 - \mathrm{i}\overline{M}_Z\overline{\Gamma}_Z. \tag{59b}$$

In addition, a very popular parameterization is the on-the-mass-shell definition of (M_Z, Γ_Z), obtained from the propagator definition in the on-the-mass-shell renormalization scheme [33]:

$$a_j^{\text{res}}(\mathsf{s}) = \frac{R_Z}{\mathsf{s} - M_Z^2 + \mathrm{i}\frac{\mathsf{s}}{M_Z}\Gamma_Z}. \tag{60}$$

The values for these different parameterizations (M, Γ) are obtained from fits of $a_j(\mathsf{s}) = a_j^{\text{res}}(\mathsf{s}) + B_j(\mathsf{s})$ to the cross section data ('line shape') $|a_j(\mathsf{s})|^2$. ($B(\mathsf{s})$ is a slowly varying background.) Depending upon the different choices (59) and (60), this leads to the different 'experimental' values of the resonance mass (M_Z and $\overline{M}_Z$ are listed in PDG book) as given in table 1: the situation for the other well-measured (hadron) resonances Δ and ρ is similar. Therefore the question is: which is the correct pair of mass and width? Or is there no right value—is the value of M and Γ just a matter of convention for the parameterization of the complex value s_R?

Using, in analogy to Wigner's definition for stable relativistic particles, the definition by causal relativistic spacetime transformations as the definition for the mass of a relativistic resonance, one can fix the values of M and Γ in the parameterization (59) uniquely and exclude the parameterization (60). From the time evolution of a relativistic Gamow vector of the semigroup representation $[\mathsf{s}_R, j]$, one calculates (for simplicity here in the rest frame $\hat{\mathbf{p}} = \mathbf{0}$)

$$H^\times|[\mathsf{s}_R, j]\hat{\mathbf{p}} = \mathbf{0} j_3^-\rangle = \sqrt{\mathsf{s}_R}|[\mathsf{s}_R, j]\hat{\mathbf{p}} = \mathbf{0} j_3^-\rangle, \tag{61}$$

where $H^\times = P_0$. If we take the parameterization (59*a*), the action of a time translation on the relativistic Gamow vector is given by

$$\phi_{\mathsf{s}_R}^G(t) = \mathrm{e}^{-\mathrm{i}H^\times t/\hbar}|[\mathsf{s}_R, j]\hat{\mathbf{p}} = \mathbf{0} j_3^-\rangle = \mathrm{e}^{-\mathrm{i}M_R t/\hbar}\mathrm{e}^{-(\Gamma_R/2)t/\hbar}|[\mathsf{s}_R, j]\hat{\mathbf{p}} = \mathbf{0} j_3^-\rangle. \tag{62}$$

The Born probability density for detecting the decay products (observable $|\psi^-\rangle\langle\psi^-|$) in the quasistable state $\phi^G_{\mathsf{s}_R,j}(t) = |Z^-\rangle$ at time $t \geqslant 0$ is then, as a consequence of (62), proportional to

$$|\langle\psi^-|\phi^G_{\mathsf{s}_R}(t)\rangle|^2 = \mathrm{e}^{-\Gamma_R t}|\langle\psi^-|\phi^G_{\mathsf{s}_R}(0)\rangle|^2. \tag{63}$$

From this we conclude that the Gamow vector with the relativistic Breit–Wigner line shape $1/(\mathsf{s}-\mathsf{s}_R)$ and the parameterization of the *S*-matrix pole position given by $\mathsf{s}_R = (M_R - \Gamma_R/2)^2$ has the lifetime $\tau = \hbar/\Gamma_R$. Therefore (M_R, Γ_R) is the correct definition of (M, Γ) for a relativistic resonance. With this, the 'right' values of mass and width of the *Z*-boson are

$$M_R = \quad \mathrm{Re}\sqrt{\mathsf{s}_R} = 91.1611 \pm 0.0023 \text{ GeV} = M_Z - 0.0026 \text{ GeV} \tag{64}$$

$$\Gamma_R = -2\,\mathrm{Im}\sqrt{\mathsf{s}_R} = 2.4943 \pm 0.0024 \text{ GeV}. \tag{65}$$

9. Conclusion

The Hardy space axiom is a refinement of the standard axiom of quantum mechanics. Standard Hilbert space quantum mechanics works fine for spectra and structure of microphysical systems in stationary states. For dynamically evolving states, for resonance scattering and for decaying states, the Hardy space axiom works better and provides a theory that unifies resonance and decay phenomena. If one admits more general operators for observables and states (as mentioned in the epilogue) one also obtains exponentially decaying states that are associated with *S*-matrix poles of order $\mathcal{N} > 1$, but these states cannot be described by vectors or kets. They are described by non-diagonalizable operators (A.6), (A.7), and their Hamiltonians contain non-diagonalizable Jordan matrices. The Hardy space axiom also introduces a novel concept: a quantum mechanical beginning of time or the semigroup time $t_0 = 0$. This concept is not entirely new. In Gell-Mann and Hartle's quantum theory of the universe [34], this $t_0 = t_{\text{big bang}}$. But how does one see this time $t_0 = 0$ in the usual experiments with quantum systems in the lab? The discussion of this question needs to be postponed to another paper.

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Epilogue

In order to be as intelligible as possible, the talk at QTS-5 was confined to the simplest cases possible: the observables were represented by vectors ψ^-, i.e., by $\Lambda = |\psi^-\rangle\langle\psi^-|$; the states were represented by vectors ϕ^+, and for the decaying states we used the Gamow state vector ϕ^G of (44). However observables are in general represented not by $|\psi^-\rangle$ but by operators that obey the Heisenberg equation (7*a*), and states are in general described by density operators W obeying the von-Neumann equation (8*a*). Some of these generalizations are accommodated in a straightforward way, e.g., the observable one generalizes $|\psi^-\rangle\langle\psi^-| \longrightarrow \Lambda = \int \mathrm{d}E\lambda(E)|^-E\rangle\langle E^-|$. Or for the state one goes to density operators $|\phi^+\rangle\langle\phi^+| \longrightarrow W$ so that the Born probabilities (14) become $|\langle\psi^-|\phi^+(t)\rangle|^2 \longrightarrow \mathrm{Tr}(\Lambda W(t))$. This can be done when the Hamiltonian H can be diagonalized, as is always the case for self-adjoint H in the

Schwartz space with the basis system (33*a*), (33*b*) in (32). The Dirac basis vector expansion in the matrix representation (33) is given as

$$\begin{pmatrix} \langle\psi|H^\times|E_1) \\ \langle\psi|H^\times|E_2) \\ \vdots \\ \langle\psi|H^\times|E_N) \\ \langle\psi|H^\times|E^-\rangle \end{pmatrix} = \begin{pmatrix} E_1 & 0 & \cdots & \cdots & 0 \\ 0 & E_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & E_n & 0 \\ 0 & 0 & \cdots & 0 & (E) \end{pmatrix} \begin{pmatrix} \langle\psi|E_1) \\ \langle\psi|E_2) \\ \vdots \\ \langle\psi|E_N) \\ \langle\psi|E^-\rangle \end{pmatrix} \tag{A.1}$$

where (E) denotes a continuously infinite submatrix, and E takes the values $0 \leqslant E < +\infty$ in the diagonal with the off-diagonal elements being zero. If one uses instead of the Hilbert space axiom (9) with Dirac expansion (32) the Hardy space axiom (49), (50), the energy wavefunctions $\langle\psi^-|E^-\rangle \in \mathcal{H} \cap \mathcal{S}$ can be continued into the lower complex energy plane (2nd sheet), and Gamow vectors (44)—generalized eigenvectors with complex generalized eigenvalue $z_R = E_R - \mathrm{i}\Gamma/2$—can appear. The Gamow vector (44) is associated with a first-order pole of the *S*-matrix (2nd sheet) at z_R [17]. Assume that there are two resonances, at z_{R_1} and z_{R_2}, and assume there are no discrete energy eigenvectors (no bound state) with $E_1, E_2, \ldots, E_N$ in (32). Then the matrix representation of the self-adjoint Hamiltonian has the following form:

$$\begin{pmatrix} \langle H\psi^-|z_{R_1}^-\rangle \\ \langle H\psi^-|z_{R_2}^-\rangle \\ \langle H\psi^-|E^-\rangle \end{pmatrix} = \begin{pmatrix} \langle\psi^-|H^\times|z_{R_1}^-\rangle \\ \langle\psi^-|H^\times|z_{R_2}^-\rangle \\ \langle\psi|H^\times|E^-\rangle \end{pmatrix} = \begin{pmatrix} z_{R_1} & 0 & 0 \\ 0 & z_{R_2} & 0 \\ 0 & 0 & (E) \end{pmatrix} \begin{pmatrix} \langle\psi^-|z_{R_1}^-\rangle \\ \langle\psi^-|z_{R_2}^-\rangle \\ \langle\psi^-|E^-\rangle \end{pmatrix} \tag{A.2}$$

where (E) denotes again a continuously infinite submatrix with real values in the diagonal. This—with one resonance at z_R—is the case we have restricted ourselves to in the main part of the paper. But the mathematical theory we have devised for a Breit–Wigner resonance provides much more.

Because the Hardy space Φ_+ is contained in the Schwartz space Φ, its dual $\Phi_+^\times$ is much richer than $\Phi^\times$: in addition to the Lippmann–Schwinger kets (40) with real energy E and their analytic continuations $|z^-\rangle$—which are tacitly assumed in any analytic *S*-matrix theory—and in addition to the ordinary Gamow kets (44), the space $\Phi_+^\times$ also contains $\mathcal{N}$-dimensional subspaces $\mathcal{M}_{z_\mathcal{N}} \subset \Phi_+^\times$ where $\mathcal{N}$ can be $\mathcal{N} = 1, 2, \ldots,$ any finite number. The subspace $\mathcal{M}_{z_\mathcal{N}}$ is spanned by $\mathcal{N}$ Jordan vectors with complex generalized eigenvalue $z_\mathcal{N} = (E_\mathcal{N} - \mathrm{i}\Gamma_\mathcal{N}/2)$,

$$|z_\mathcal{N}^- \succ^{(0)}, |z_\mathcal{N}^- \succ^{(1)}, \ldots, |z_\mathcal{N}^- \succ^{(k)}, \ldots, |z_\mathcal{N}^- \succ^{(\mathcal{N}-1)}, \tag{A.3}$$

(where $\mathcal{M}_{z_{\mathcal{N}=1}}$ is the space spanned by the ordinary Gamow ket in (44) $|z_1^-\rangle^{(0)} = |E_R - \mathrm{i}\Gamma/2^-\rangle$). The kth-order Gamow vector $|z_\mathcal{N}^- \succ^{(k)}$, $k = 0, 1, \ldots, (\mathcal{N} - 1)$, is a Jordan vector of degree $(k + 1)$, i.e., it fulfils the generalized eigenvalue equations [35, 36]

$$\begin{aligned} &(H^\times - z_\mathcal{N})^{k+1}|z_\mathcal{N}^- \succ^{(k)} = 0; \\ &H^\times|z_\mathcal{N}^- \succ^{(0)} = z_\mathcal{N}|z_\mathcal{N}^- \succ^{(0)}; \\ &H^\times|z_\mathcal{N}^- \succ^{(k)} = z_\mathcal{N}|z_\mathcal{N}^- \succ^{(k)} + \Gamma_\mathcal{N}|z_\mathcal{N}^- \succ^{(k-1)} \qquad \text{for} \quad k = 1, 2, \ldots, (\mathcal{N} - 1). \end{aligned} \tag{A.4}$$

These equations are, like the eigenvector equation for Dirac kets (33*a*) and for Gamow vectors (44) (Gamow vector = Jordan vectors of degree 1), understood as generalized eigenvector equations (i.e., functionals) over the space Φ_+. Jordan block matrices for non-Hermitian Hamiltonians have been discussed before, e.g. [37, 38], and were used for finite-dimensional phenomenological expressions of the *S*-matrix [39–42], but could not be implemented in the general framework of quantum mechanics using the Hilbert space or the Schwartz space axiom. Using the Hardy space axiom (49), (50) the Jordan–Gamow vectors can be derived from the $\mathcal{N}$th-order pole of the unitary S-matrix [43, 36] in very much the same way as the ordinary Gamow vectors were derived from the first-order S-matrix pole (43) [17]. The matrix of the Hamiltonian $H^\times$ has in the diagonal the complex eigenvalues $z_R = E_R - \mathrm{i}\Gamma_R/2$ of the first-order pole position for the ordinary Gamow kets. In addition, it contains Jordan blocks of Jordan–Gamow kets. For example, in the case of two first-order poles at z_{R_1} and z_{R_2} and one second-order pole at $z_2 = E_2 - \mathrm{i}\Gamma_2/2$ the matrix of the Hamiltonian is given by

$$\begin{pmatrix} \langle H\psi^-|z_2^- \succ^{(0)} \\ \langle H\psi^-|z_2^- \succ^{(1)} \\ \langle H\psi^-|z_{R_1}\rangle \\ \langle H\psi^-|z_{R_2}\rangle \\ \langle H\psi^-|E\rangle \end{pmatrix} = \begin{pmatrix} \langle \psi^-|H^\times|z_2^- \succ^{(0)} \\ \langle \psi^-|H^\times|z_2^- \succ^{(1)} \\ \langle \psi^-|H^\times|z_{R_1}\rangle \\ \langle \psi^-|H^\times|z_{R_2}\rangle \\ \langle \psi^-|H^\times|E\rangle \end{pmatrix} = \begin{pmatrix} z_2 & 0 & & & \\ \Gamma_2 & z_2 & & & \\ & & z_{R_1} & & \\ & & & z_{R_2} & \\ & & & & (E) \end{pmatrix} \begin{pmatrix} \langle \psi^-|z_2^- \succ^{(0)} \\ \langle \psi^-|z_2^- \succ^{(1)} \\ \langle \psi^-|z_{R_1}\rangle \\ \langle \psi^-|z_{R_2}\rangle \\ \langle \psi^-|E\rangle \end{pmatrix}. \tag{A.5}$$

Here (E) denotes the continuously infinite matrix with diagonal elements E, as in (A.2). Each z_{R_i} corresponds to the ordinary Gamow ket (44). And the 2×2 matrix on the top left corner is the Jordan block (A.4) of degree 2.

In general, from the $\mathcal{N}$th-order S-matrix pole at $z_\mathcal{N}$ one obtains the $\mathcal{N}$ basis vectors (A.3) and an $\mathcal{N}$-dimensional Jordan block (A.4) in place of the two-dimensional Jordan block in the matrix (A.5). This means the second-order or $\mathcal{N}$th-order pole of the S-matrix can no longer be described by a state vector, like the bound states by $|E_n)$ with real discrete eigenvalues, or the first-order resonance states (Gamow states) by $|z_{R_i}^-\rangle$ with complex eigenvalue z_{R_i} of the self-adjoint Hamiltonian H. Instead, the state derived from the $\mathcal{N}$th-order S-matrix pole is described by a non-diagonalizable density operator or state operator [36]

$$W_{\mathrm{PT}} = 2\pi\Gamma \sum_{n=0}^{\mathcal{N}-1} \binom{\mathcal{N}}{n+1} (-i)^n W^{(n)} \tag{A.6}$$

where the operators $W^{(n)}$ are defined as

$$W^{(n)} = \sum_{k=0}^{n} |z_\mathcal{N}^- \succ^{(k)(n-k)}\prec^- z_\mathcal{N}|, \qquad n = 0, 1, 2, \ldots, \mathcal{N}-1. \tag{A.7}$$

The pole term of the $\mathcal{N}$th-order S-matrix is associated with a sum (A.6) of the operators $W^{(n)}$. The operators $W^{(n)}$ represent components of this pole term state W_{PT} which are 'irreducible' in a way specified below in (A.11). In the case $\mathcal{N} = 1$ (ordinary first-order resonance pole), the operator (A.6) becomes

$$W_{\mathrm{PT}} = 2\pi\Gamma|z_1^- \succ^{(0)(0)}\prec^- z_1| = 2\pi\Gamma W^{(0)} = |\phi^G\rangle\langle\phi^G|. \tag{A.8}$$

This is the operator description of the Gamow state whose vector description is given by ϕ^G of (44) and whose time evolution is, in agreement with (45), given by

$$\begin{aligned} W^G(t) &= \mathrm{e}^{-\mathrm{i}H^\times t}|\phi^G\rangle\langle\phi^G|\mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{-\mathrm{i}z_R t}|\phi^G\rangle\langle\phi^G|\mathrm{e}^{\mathrm{i}z_R^* t} \\ &= \mathrm{e}^{-\mathrm{i}(E_R - \mathrm{i}\Gamma/2)t}|\phi^G\rangle\langle\phi^G|\mathrm{e}^{\mathrm{i}(E_R+\mathrm{i}\Gamma/2)t} = \mathrm{e}^{-\Gamma t}W^G(0). \end{aligned} \tag{A.9}$$

Only the 0th-order Gamow vector $|z^- \succ^{(0)} = \frac{1}{\sqrt{2\pi\Gamma}}\phi^G$ has exponential time evolution. In general, the kets $|z_\mathcal{N}^- \succ^{(k)}$, $k = 0, \ldots, \mathcal{N}-1$ have very complicated time evolution given by

$$\mathrm{e}^{-\mathrm{i}H^\times t}|z_\mathcal{N}^- \succ^{(k)} = \mathrm{e}^{-\mathrm{i}z_\mathcal{N} t} \sum_{\nu=0}^{k} \frac{\Gamma^\nu}{\nu!}(-it)^\nu |z_\mathcal{N}^- \succ^{(k-\nu)} \qquad t \geqslant 0. \tag{A.10}$$

These are representations of the time translation semigroup, which (for $\mathcal{N} > 1$) are not one dimensional. The existence of this kind of representation for the causal spacetime translation group has already been mentioned in [16]. The appearance of a linear term in the time dependence for a second-order pole resonance $\mathcal{N} = 1$ in (A.10) has been well known for many years [22, 44, 45]. That such linear time dependence has never been observed was used as an argument against the existence of higher order pole resonances [45]. This was a misconception because the state associated with the S-matrix pole is described by a state operator (density matrix) (A.6), which has an exponential time evolution, as we shall report now.

The density operator or statistical operator of the state *derived* from the $\mathcal{N}$th-order pole is given by (A.6) with (A.7). It is remarkable that with the use of (A.10) one obtains after a complicated calculation a very simple result

$$W^{(n)}(t) = \mathrm{e}^{-\mathrm{i}H^\times t} W^{(n)} \mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{-\Gamma t} \sum_{k=0}^{n} |z^- \succ^{(k)(k-n)} \prec^- z| = \mathrm{e}^{-\Gamma t} W^{(n)}(0), \qquad t \geqslant 0. \tag{A.11}$$

This result means that the complicated non-reducible (i.e., 'mixed') microphysical state operator $W^{(n)}$ defined by (A.6) and (A.7) has a simple and purely exponential semigroup time evolution, like the 0th-order Gamow state (A.9), and thus leads to the exponential law for the probabilities. The operators $W^{(n)}$ are probably the only operators formed by the dyadic products $|z_{\mathcal{N}}^- \succ^{(m)} {}^{(\ell)} \prec^- z_{\mathcal{N}}|$ with $m, \ell = 0, 1, \ldots, n$, that have purely exponential time evolution. Thus the operators $W^{(n)}, n = 1, 2, \ldots, \mathcal{N} - 1$, are distinguished from all other operators in $\mathcal{M}_{z_{\mathcal{N}}}$. The microphysical decaying state operator associated with the $\mathcal{N}$th-order pole of the unitary S-matrix W_{PT} is the sum (A.6) of these $W^{(n)}$. Because of the result (A.11) (independent of the time evolution of n) this sum has again the simple, exponential time evolution

$$W_{\mathrm{PT}}(t) \equiv \mathrm{e}^{-\mathrm{i}H^\times t} W_{\mathrm{PT}}\, \mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{-\Gamma_{\mathcal{N}} t} W_{\mathrm{PT}}, \qquad t \geqslant 0. \tag{A.12}$$

Thus, the theory that describes exponentially decaying states by Gamow vectors (44) also admits the possibility of exponentially decaying states that are associated with S-matrix poles of $\mathcal{N}$th order at the position $z_{\mathcal{N}} = E_{\mathcal{N}} - \mathrm{i}\Gamma_{\mathcal{N}}/2$. The 'mixed' state (A.6) associated with the $\mathcal{N}$th-order S-matrix pole also has an exponential time evolution. The probability to register an observable $\Lambda(t)$ (representing, e.g., a detector) in the state $W^{(n)}$ or W_{PT} is obtained using (A.11) or (A.12) as

$$\begin{aligned} \mathrm{Tr}(\Lambda(t) W_{\mathrm{PT}}) &= \mathrm{Tr}(\mathrm{e}^{\mathrm{i}Ht} \Lambda\, \mathrm{e}^{-\mathrm{i}H^\times t}\, W_{\mathrm{PT}}) = \mathrm{Tr}(\Lambda\, \mathrm{e}^{-\mathrm{i}Ht}\, W_{\mathrm{PT}}\, \mathrm{e}^{\mathrm{i}Ht}) = \mathrm{e}^{-\Gamma_{\mathcal{N}} t}\, \mathrm{Tr}(\Lambda W_{\mathrm{PT}}), \\ & 0 < t < \infty. \end{aligned} \tag{A.13}$$

These exponentially decaying states cannot be described by a vector like $|\phi^G\rangle\langle\phi^G|$. The simplest choice for this kind of state operator is the one associated with the pole term of a second-order S-matrix pole at z_2:

$$\begin{aligned} W_{\mathrm{PT}} &= 2\pi\Gamma(2W^{(0)} - \mathrm{i}W^{(1)}) \\ &= 2\pi\Gamma(2|z_2^- \succ^{(0)\,(0)} \prec^- z_2| - \mathrm{i}(|z_2^- \succ^{(0)\,(1)} \prec^- z_2| + |z_2^- \succ^{(1)\,(0)} \prec^- z_2|)). \end{aligned} \tag{A.14}$$

In the subspaces $\mathcal{M}_{z_{\mathcal{N}}} \subset \Phi_+^\times$ associated with the $\mathcal{N}$th-order pole, the Hamiltonian H is non-diagonalizable and so is the state operator. The Hardy space theory, which was *needed* for the theoretical description of first-order pole states (by Gamow vectors), also provides the mathematical means for higher order pole states described by non-diagonalizable operators; this is not possible in the Hilbert space or the Schwartz space. This does not constitute a proof that these states exist in nature—higher order S-matrix poles may be excluded for some other physical reasons—but it provides a mathematical possibility, and it is an argument against the exclusion [45] of exponentially decaying higher order resonance states.

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